

ANALOGUES OF THE RAMANUJAN–MORDELL THEOREM

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ABSTRACT. The Ramanujan–Mordell Theorem for sums of an even number of squares is extended to other quadratic forms and quadratic polynomials.

1. INTRODUCTION

One of the classical problems in number theory is to determine exact formulas for the number of representations of a positive integer n as a sum of $2k$ squares, which we denote by $r(2k; n)$. If we set

$$(1.1) \quad z = z(\tau) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+n^2}$$

where, here and throughout the remainder of this work, τ is a complex number with positive imaginary part and $q = e^{2\pi i\tau}$, then (considering z as a power series in q)

$$\sum_{n=0}^{\infty} r(2k; n) q^n = z^k.$$

The function z^k is a modular form and it is well-known that

$$z^k(\tau) = E_k^*(\tau) + C_k(\tau),$$

where $E_k^*(\tau)$ is an Eisenstein series and $C_k(\tau)$ is a cusp form. In his remarkable work, Ramanujan [24, Eqs. (145)–(147)] stated without proof explicit formulas for $E_k^*(\tau)$ and $C_k(\tau)$, and hence deduced the value of the coefficients $r(2k; n)$. Ramanujan’s result was first proved by Mordell [20]. To state it we need Dedekind’s eta function, which is defined by

$$\eta(\tau) := q^{1/24} \prod_{j=1}^{\infty} (1 - q^j).$$

Here and throughout, we write η_m for $\eta(m\tau)$ for any positive integer m .

Theorem 1.1 (Ramanujan–Mordell). *Suppose k is a positive integer. Let z be defined by (1.1). Then*

$$(1.2) \quad z^k = F_k(\tau) + z^k \sum_{1 \leq j \leq \frac{(k-1)}{4}} c_{j,k} x^j$$

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where $c_{j,k}$ are numerical constants that depend on j and k ,

$$x = x(\tau) := \frac{\eta_2^{48}}{\eta_1^{24}\eta_4^{24}},$$

and $F_k(\tau)$ is an Eisenstein series defined by:

$$F_1(\tau) := 1 + 4 \sum_{j=1}^{\infty} \frac{q^j}{1 + q^{2j}},$$

and for $k \geq 1$,

$$F_{2k}(\tau) := 1 - \frac{4k(-1)^k}{(2^{2k} - 1)\mathcal{B}_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1}q^j}{1 - (-1)^{k+j}q^j},$$

and

$$F_{2k+1}(\tau) := 1 + \frac{4(-1)^k}{\mathcal{E}_{2k}} \sum_{j=1}^{\infty} \left(\frac{(2j)^{2k}q^j}{1 + q^{2j}} - \frac{(-1)^{k+j}(2j-1)^{2k}q^{2j-1}}{1 - q^{2j-1}} \right).$$

Here \mathcal{B}_k and \mathcal{E}_k are the Bernoulli numbers and Euler numbers, respectively, defined by

$$\frac{u}{e^u - 1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}_k}{k!} u^k \quad \text{and} \quad \frac{1}{\cosh u} = \sum_{k=0}^{\infty} \frac{\mathcal{E}_k}{k!} u^k.$$

The reader is referred to [9, p. 2] for a brief account of the history of the study of Theorem 1.1 for various k .

The goal of this work is to prove the analogues of the Ramanujan–Mordell Theorem for which the quadratic form $m^2 + n^2$ in (1.1) is replaced with the quadratic form $m^2 + pn^2$, or by the quadratic polynomial

$$\frac{m(m+1)}{2} + p \frac{n(n+1)}{2},$$

where $p \in \{3, 7, 11, 23\}$.

This work is organized as follows. In Section 2, we set up definitions, state the main results and explicate several examples. Proofs of the main results are given in Section 3.

2. DEFINITIONS AND MAIN RESULTS

For $k \geq 1$, define the normalized Eisenstein series by

$$(2.1) \quad E_{2k}(\tau) := 1 - \frac{4k}{\mathcal{B}_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1}q^j}{1 - q^j}.$$

Let p be an odd prime. The generalized Bernoulli numbers $\mathcal{B}_{k,p}$ are defined by

$$(2.2) \quad \frac{x}{e^{px} - 1} \sum_{j=1}^{p-1} \chi_p(j) e^{jx} = \sum_{k=0}^{\infty} \mathcal{B}_{k,p} \frac{x^k}{k!},$$

where $\chi_p(j) := \left(\frac{j}{p}\right)$ is the Legendre symbol. Let k be a positive integer which satisfies

$$k \equiv \frac{p-1}{2} \pmod{2}.$$

The generalized Eisenstein series $E_k^0(\tau; \chi_p)$ and $E_k^\infty(\tau; \chi_p)$ at the cusps 0 and $i\infty$, respectively, are defined by

$$E_k^0(\tau; \chi_p) := \delta_{k,1} - \frac{2k}{\mathcal{B}_{k,p}} \sum_{j=1}^{\infty} \frac{j^{k-1}}{1 - q^{pj}} \sum_{\ell=1}^{p-1} \left(\frac{\ell}{p}\right) q^{j\ell}$$

and

$$E_k^\infty(\tau; \chi_p) := 1 - \frac{2k}{\mathcal{B}_{k,p}} \sum_{j=1}^{\infty} \left(\frac{j}{p}\right) \frac{j^{k-1} q^j}{1 - q^j},$$

where $\delta_{m,n}$ is the Kronecker delta function, defined by

$$\delta_{m,n} := \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Certain linear combinations of Eisenstein series and generalized Eisenstein series will occur in the main results, and in anticipation of this we define series $G_k(\tau; p)$, $\tilde{G}_{2k+1}(\tau; p)$, $F_k(\tau; p)$ and $\tilde{F}(\tau; p)$ as follows. For $k \geq 1$, let

$$G_{2k}(\tau; p) := E_{2k}(\tau) + (-p)^k E_{2k}(p\tau).$$

For $k \geq 0$ let

$$G_{2k+1}(\tau; p) := E_{2k+1}^\infty(\tau; \chi_p) + (-p)^k E_{2k+1}^0(\tau; \chi_p)$$

and

$$\tilde{G}_{2k+1}(\tau; p) := E_{2k+1}^\infty(\tau; \chi_p) - (-p)^k E_{2k+1}^0(\tau; \chi_p).$$

For $p = 3$ or 11 and $k \geq 0$, let

$$(2.3) \quad F_{2k+1}(\tau; p) := \frac{G_{2k+1}(\tau; p) + 2^{2k+1} G_{2k+1}(4\tau; p)}{(2^{2k} + 1)(1 + \delta_{k,0})}$$

and

$$(2.4) \quad \tilde{F}_{2k+1}(\tau; p) := \frac{(2^{2k+1} - 2)\tilde{G}_{2k+1}(\tau; p) - 2^{2k+1} G_{2k+1}(2\tau; p) + 2G_{2k+1}(\tau/2; p)}{2^{4k+2}(2^{2k+1} + 1)(1 + \delta_{k,0})}.$$

For $p = 7$ or 23 and $k \geq 0$, let

$$(2.5) \quad F_{2k+1}(\tau; p) := \frac{G_{2k+1}(\tau; p) - 2G_{2k+1}(2\tau; p) + 2^{2k+1} G_{2k+1}(4\tau; p)}{(2^{2k+1} - 1)(1 + \delta_{k,0})}$$

and

$$(2.6) \quad \tilde{F}_{2k+1}(\tau; p) := \frac{G_{2k+1}(\tau; p) - G_{2k+1}(2\tau; p)}{2^{2k+1}(2^{2k+1} - 1)(1 + \delta_{k,0})}.$$

For $p = 3, 7, 11$ or 23 and any integer $k \geq 1$, let

$$(2.7) \quad F_{2k}(\tau; p) := \frac{G_{2k}(\tau; p) - 2G_{2k}(2\tau; p) + 2^{2k} G_{2k}(4\tau; p)}{(2^{2k} - 1)(1 + (-p)^k)}$$

and

$$(2.8) \quad \tilde{F}_{2k}(\tau; p) := \frac{G_{2k}(\tau; p) - G_{2k}(2\tau; p)}{2^{2k}(2^{2k} - 1)(1 + (-p)^k)}.$$

The observant reader will notice that the series \tilde{G}_{2k+1} occurs only once in the definitions (2.3)–(2.8), and that is as one of the terms in (2.4). A reason for this will be seen later in Lemma 3.3, in which (3.6) and (3.7) imply that for $p = 3$ or 11

$$G_{2k+1}\left(\tau + \frac{1}{2}; p\right) = -G_{2k+1}(\tau; p) + \left(2 - 2^{2k+1}\right) \tilde{G}_{2k+1}(2\tau; p) + 2^{2k+1} G_{2k+1}(4\tau; p).$$

For $p = 3, 7, 11$ or 23 , let z_p and x_p be defined by

$$(2.9) \quad z_p = z_p(\tau) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2 + pn^2},$$

and

$$(2.10) \quad x_p = x_p(\tau) := \frac{(\eta_1 \eta_4 \eta_p \eta_{4p})^{24/(p+1)}}{(\eta_2 \eta_{2p})^{48/(p+1)}}.$$

The analogue of the Ramanujan–Mordell Theorem, and the main result of this work, is:

Theorem 2.1. *Suppose $p = 3, 7, 11$ or 23 and let k be a positive integer. Then*

$$z_p^k = F_k(\tau; p) + z_p^k \sum_{1 \leq j < \frac{(p+1)k}{8}} c_{p,k,j} x_p^j,$$

where $c_{p,k,j}$ are numerical constants that depend only on p, k , and j .

There is an analogue of the Ramanujan–Mordell theorem that involves sum of triangular numbers instead of sums of squares; that is, replace the quadratic form $m^2 + n^2$ in (1.1) and Theorem 1.1 with the quadratic polynomial $m(m+1)/2 + n(n+1)/2$. See [24, pp. 190–191] and [10, Theorems 3.5 and 3.6]. To state the corresponding analogue of Theorem 2.1, let \tilde{z}_p and \tilde{x}_p be defined by

$$(2.11) \quad \tilde{z}_p = \tilde{z}_p(\tau) := q^{(p+1)/8} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{\frac{m(m+1)}{2} + p \frac{n(n+1)}{2}}$$

and

$$(2.12) \quad \tilde{x}_p = \tilde{x}_p(\tau) := \left(\frac{\eta_1 \eta_p}{\eta_2 \eta_{2p}} \right)^{24/(p+1)}.$$

Corollary 2.2. *Suppose $p = 3, 7, 11$ or 23 and let k be a positive integer. Then*

$$\tilde{z}_p^k = \tilde{F}_k(\tau; p) + \tilde{z}_p^k \sum_{1 \leq j < \frac{(p+1)k}{8}} \tilde{c}_{p,k,j} \tilde{x}_p^j,$$

where $\tilde{c}_{p,k,j}$ are numerical constants that depend only on p, k , and j and are related to the $c_{p,k,j}$ in Theorem 2.1 by $\tilde{c}_{p,k,j} = 2^{-24j/(p+1)} (-1)^j c_{p,k,j}$.

In the remainder of this section we describe some special cases of Theorem 2.1 and Corollary 2.2. Let $\varphi(q)$ and $\psi(q)$ be Ramanujan’s theta functions defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Because of the occurrence of the argument $\tau/2$ in one of the terms in (2.4), we have replaced τ with 2τ , and hence q with q^2 , to obtain the examples (2.14), (2.18) and (2.24), below.

Example 2.3. For $k = 1$ and $p = 3, 7, 11$ or 23 , Theorem 2.1 and Corollary 2.2 give

$$(2.13) \quad \varphi(q)\varphi(q^3) = 1 + 2 \sum_{j=1}^{\infty} \left(\frac{j}{3} \right) \left(\frac{q^j}{1-q^j} + \frac{2q^{4j}}{1-q^{4j}} \right),$$

$$(2.14) \quad q\psi(q^2)\psi(q^6) = \sum_{j=1}^{\infty} \left(\frac{j}{3} \right) \left(\frac{q^j}{1-q^j} - \frac{q^{4j}}{1-q^{4j}} \right),$$

$$(2.15) \quad \varphi(q)\varphi(q^7) = 1 + 2 \sum_{j=1}^{\infty} \left(\frac{j}{7} \right) \left(\frac{q^j}{1-q^j} - \frac{2q^{2j}}{1-q^{2j}} + \frac{2q^{4j}}{1-q^{4j}} \right),$$

$$(2.16) \quad q\psi(q)\psi(q^7) = \sum_{j=1}^{\infty} \left(\frac{j}{7} \right) \left(\frac{q^j}{1-q^j} - \frac{q^{2j}}{1-q^{2j}} \right),$$

$$(2.17) \quad \varphi(q)\varphi(q^{11}) = 1 + \frac{2}{3} \sum_{j=1}^{\infty} \left(\frac{j}{11} \right) \left(\frac{q^j}{1-q^j} + \frac{2q^{4j}}{1-q^{4j}} \right) + \frac{4}{3} \eta_2 \eta_{22},$$

$$(2.18) \quad q^3\psi(q^2)\psi(q^{22}) = \frac{1}{3} \sum_{j=1}^{\infty} \left(\frac{j}{11} \right) \left(\frac{q^j}{1-q^j} - \frac{q^{4j}}{1-q^{4j}} \right) - \frac{1}{3} \eta_2 \eta_{22},$$

$$(2.19) \quad \begin{aligned} \varphi(q)\varphi(q^{23}) &= 1 + \frac{2}{3} \sum_{j=1}^{\infty} \left(\frac{j}{23} \right) \left(\frac{q^j}{1-q^j} - \frac{2q^{2j}}{1-q^{2j}} + \frac{2q^{4j}}{1-q^{4j}} \right) \\ &\quad + \frac{4}{3} \frac{\eta_2^3 \eta_{46}^3}{\eta_1 \eta_4 \eta_{23} \eta_{92}} - \frac{4}{3} \eta_2 \eta_{46}, \end{aligned}$$

$$(2.20) \quad q^3\psi(q)\psi(q^{23}) = \frac{1}{3} \sum_{j=1}^{\infty} \left(\frac{j}{23} \right) \left(\frac{q^j}{1-q^j} - \frac{q^{2j}}{1-q^{2j}} \right) - \frac{1}{3} \eta_1 \eta_{23} - \frac{2}{3} \eta_2 \eta_{46}.$$

The identity (2.13) was first stated in an equivalent form by Lorenz [19, p. 420]. Both (2.13) and (2.14) were given by Ramanujan in his second notebook [23, Ch. 19, Entry 3]. See Berndt [8, pp. 223–224], Fine [13, p. 73, (31.16), (31.22)] and Hirschhorn [14] for proofs and further information.

The identities (2.15) and (2.16) also appear in Ramanujan's second notebook [23, Ch. 19, Entry 17] and proofs have been given by Berndt [8, pp. 302–304]. The identity (2.15) has also been proved by Pall [21].

The identities (2.17) and (2.19) have been recently proved by the third author [25].

The identities (2.18) and (2.20) are new.

Example 2.4. For $p = 3$, the cases $k = 2, 3, 4$ and 6 of Theorem 2.1 and Corollary 2.2 give

$$(2.21) \quad \varphi^2(q)\varphi^2(q^3) = 1 + 4 \sum_{j=1}^{\infty} \left(\frac{jq^j}{1-q^j} - \frac{2jq^{2j}}{1-q^{2j}} - \frac{3jq^{3j}}{1-q^{3j}} + \frac{4jq^{4j}}{1-q^{4j}} + \frac{6jq^{6j}}{1-q^{6j}} - \frac{12jq^{12j}}{1-q^{12j}} \right),$$

$$(2.22) \quad q\psi^2(q)\psi^2(q^3) = \sum_{j=1}^{\infty} \left(\frac{jq^j}{1-q^j} - \frac{jq^{2j}}{1-q^{2j}} - \frac{3jq^{3j}}{1-q^{3j}} + \frac{3jq^{6j}}{1-q^{6j}} \right),$$

$$(2.23) \quad \varphi^3(q)\varphi^3(q^3) = 1 + 3 \sum_{j=1}^{\infty} \left(\frac{j^2(q^j - q^{2j})}{1-q^{3j}} + \frac{8j^2(q^{4j} - q^{8j})}{1-q^{12j}} \right)$$

$$\begin{aligned}
& - \sum_{j=1}^{\infty} \left(\frac{j}{3} \right) \left(\frac{j^2 q^j}{1-q^j} + \frac{8j^2 q^{4j}}{1-q^{4j}} \right) + 4\eta_2^3 \eta_6^3, \\
(2.24) \quad & q^3 \psi^3(q^2) \psi^3(q^6) = \frac{3}{32} \sum_{j=1}^{\infty} \left(\frac{j^2(q^j - q^{2j})}{1-q^{3j}} - \frac{3j^2(q^{2j} - q^{4j})}{1-q^{6j}} - \frac{4j^2(q^{4j} - q^{8j})}{1-q^{12j}} \right) \\
& - \frac{1}{32} \sum_{j=1}^{\infty} \left(\frac{j}{3} \right) \left(\frac{j^2 q^j}{1-q^j} + \frac{3j^2 q^{2j}}{1-q^{2j}} - \frac{4j^2 q^{4j}}{1-q^{4j}} \right) - \frac{1}{16} \eta_2^3 \eta_6^3, \\
(2.25) \quad &
\end{aligned}$$

$$\begin{aligned}
\varphi^4(q) \varphi^4(q^3) &= 1 + \frac{8}{5} \sum_{j=1}^{\infty} \left(\frac{j^3 q^j}{1-q^j} - \frac{2j^3 q^{2j}}{1-q^{2j}} + \frac{9j^3 q^{3j}}{1-q^{3j}} + \frac{16j^3 q^{4j}}{1-q^{4j}} - \frac{18j^3 q^{6j}}{1-q^{6j}} + \frac{144j^3 q^{12j}}{1-q^{12j}} \right) \\
& + \frac{32}{5} \frac{\eta_2^8 \eta_6^8}{\eta_1^2 \eta_3^2 \eta_4^2 \eta_{12}^2}, \\
(2.26) \quad &
\end{aligned}$$

$$\begin{aligned}
q^2 \psi^4(q) \psi^4(q^3) &= \frac{1}{10} \sum_{j=1}^{\infty} \left(\frac{j^3 q^j}{1-q^j} - \frac{j^3 q^{2j}}{1-q^{2j}} + \frac{9j^3 q^{3j}}{1-q^{3j}} - \frac{9j^3 q^{6j}}{1-q^{6j}} \right) - \frac{1}{10} \eta_1^2 \eta_2^2 \eta_3^2 \eta_6^2, \\
(2.27) \quad &
\end{aligned}$$

$$\begin{aligned}
\varphi^6(q) \varphi^6(q^3) &= 1 + \frac{4}{13} \sum_{j=1}^{\infty} \left(\frac{j^5 q^j}{1-q^j} - \frac{2j^5 q^{2j}}{1-q^{2j}} - \frac{27j^5 q^{3j}}{1-q^{3j}} + \frac{64j^5 q^{4j}}{1-q^{4j}} + \frac{54j^5 q^{6j}}{1-q^{6j}} - \frac{1728j^5 q^{12j}}{1-q^{12j}} \right) \\
& + \frac{152}{13} \frac{\eta_2^{18} \eta_6^{18}}{\eta_1^6 \eta_3^6 \eta_4^6 \eta_{12}^6} - \frac{256}{13} \eta_2^6 \eta_6^6, \\
(2.28) \quad &
\end{aligned}$$

$$\begin{aligned}
q^3 \psi^6(q) \psi^6(q^3) &= \frac{1}{208} \sum_{j=1}^{\infty} \left(\frac{j^5 q^j}{1-q^j} - \frac{j^5 q^{2j}}{1-q^{2j}} - \frac{27j^5 q^{3j}}{1-q^{3j}} + \frac{27j^5 q^{6j}}{1-q^{6j}} \right) - \frac{1}{208} \eta_1^6 \eta_3^6 - \frac{19}{104} \eta_2^6 \eta_6^6.
\end{aligned}$$

The identity (2.21) was first stated without proof in an equivalent form by Liouville [17, 18]. See Pepin [22], Bachmann [5], Kloosterman [15] and Alaca et al. [2] for proofs. Both (2.21) and (2.22) appear in Ramanujan's second notebook [23, Ch. 19, Entry 3]. Proofs have been given by Fine [13, (31.4)–(31.43) and (33.2)] and Berndt [8, pp. 223–226].

A formula equivalent to (2.23) was proved by Alaca et al. in [3], where it was attributed to Berkovich and Yeřilyurt.

The identity (2.25) was proved by Alaca and Williams [4]. A formula similar to (2.25), in which the coefficients in $\eta_2^2 \eta_2^2 \eta_3^2 \eta_6^2$ are given as a quadruple sum, has been given by Beridze [6].

A formula equivalent to (2.27) was given by Alaca [1]; that formula involves three cusp forms on the right hand side, while ours involves only two¹.

The identities (2.24), (2.26) and (2.28) are believed to be new.

Example 2.5. For $p = 7$, the cases $k = 2$ and 3 of Theorem 2.1 give

$$(2.29) \quad \varphi^2(q) \varphi^2(q^7) = 1 + \frac{4}{3} \sum_{j=1}^{\infty} \left(\frac{j q^j}{1-q^j} - \frac{2j q^{2j}}{1-q^{2j}} + \frac{4j q^{4j}}{1-q^{4j}} - \frac{7j q^{7j}}{1-q^{7j}} + \frac{14j q^{14j}}{1-q^{14j}} - \frac{28j q^{28j}}{1-q^{28j}} \right)$$

¹This is because, if $b(q) = \eta_1^6 \eta_3^6$, then $b(q) + 12b(q^2) + 64b(q^4) + b(-q) = 0$.

$$\begin{aligned}
& + \frac{8}{3} \frac{\eta_2^4 \eta_{14}^4}{\eta_1 \eta_4 \eta_7 \eta_{28}}, \\
(2.30) \quad q^2 \psi^2(q) \psi^2(q^7) &= \frac{1}{3} \sum_{j=1}^{\infty} \left(\frac{j q^j}{1-q^j} - \frac{j q^{2j}}{1-q^{2j}} - \frac{7 j q^{7j}}{1-q^{7j}} + \frac{7 j q^{14j}}{1-q^{14j}} \right) \\
& - \frac{1}{3} \eta_1 \eta_2 \eta_7 \eta_{14},
\end{aligned}$$

$$\begin{aligned}
(2.31) \quad \varphi^3(q) \varphi^3(q^7) &= 1 + \frac{7}{8} \sum_{j=1}^{\infty} \frac{j^2 (q^j + q^{2j} - q^{3j} + q^{4j} - q^{5j} - q^{6j})}{1-q^{7j}} \\
& - \frac{7}{4} \sum_{j=1}^{\infty} \frac{j^2 (q^{2j} + q^{4j} - q^{6j} + q^{8j} - q^{10j} - q^{12j})}{1-q^{14j}} \\
& + 7 \sum_{j=1}^{\infty} \frac{j^2 (q^{4j} + q^{8j} - q^{12j} + q^{16j} - q^{20j} - q^{24j})}{1-q^{28j}} \\
& - \frac{1}{8} \sum_{j=1}^{\infty} \left(\frac{j}{7} \right) \left(\frac{j^2 q^j}{1-q^j} - \frac{2 j^2 q^{2j}}{1-q^{2j}} + \frac{8 j^2 q^{4j}}{1-q^{4j}} \right) \\
& + \frac{21}{4} \frac{\eta_2^9 \eta_{14}^9}{\eta_1^3 \eta_4^3 \eta_7^3 \eta_{28}^3} - 6 \eta_2^3 \eta_{14}^3,
\end{aligned}$$

$$\begin{aligned}
(2.32) \quad q^3 \psi^3(q) \psi^3(q^7) &= \frac{7}{64} \sum_{j=1}^{\infty} \frac{j^2 (q^j + q^{2j} - q^{3j} + q^{4j} - q^{5j} - q^{6j})}{1-q^{7j}} \\
& - \frac{7}{64} \sum_{j=1}^{\infty} \frac{j^2 (q^{2j} + q^{4j} - q^{6j} + q^{8j} - q^{10j} - q^{12j})}{1-q^{14j}} \\
& - \frac{1}{64} \sum_{j=1}^{\infty} \left(\frac{j}{7} \right) \left(\frac{j^2 q^j}{1-q^j} - \frac{j^2 q^{2j}}{1-q^{2j}} \right) - \frac{3}{32} \eta_1^3 \eta_7^3 - \frac{21}{32} \eta_2^3 \eta_{14}^3.
\end{aligned}$$

Identities equivalent to (2.29) and (2.30) have been proved in [12]. The identities (2.31) and (2.32) arise in the theory of 7-cores and were proved by Berkovich and Yesilyurt [7].

3. PROOFS

For any positive integer N , let us define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \pmod{N} \right\}.$$

We require the explicit modularity properties of the Eisenstein series on $\Gamma_0(p)$ as well as the Atkin-Lehner involution W_p .

Lemma 3.1. *For $p = 3, 7, 11$ or 23 , and any integer $k \geq 0$, and for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$, we have*

$$(3.1) \quad E_{2k+1}^0 \left(\frac{a\tau + b}{c\tau + d}; \chi_p \right) = \left(\frac{d}{p} \right) (c\tau + d)^{2k+1} E_{2k+1}^0(\tau; \chi_p),$$

$$(3.2) \quad E_{2k+1}^\infty \left(\frac{a\tau + b}{c\tau + d}; \chi_p \right) = \left(\frac{d}{p} \right) (c\tau + d)^{2k+1} E_{2k+1}^\infty(\tau; \chi_p),$$

$$(3.3) \quad E_{2k+1}^{\infty} \left(-\frac{1}{p\tau}; \chi_p \right) = \frac{1}{i\sqrt{p}} (p\tau)^{2k+1} E_{2k+1}^0(\tau; \chi_p),$$

$$(3.4) \quad E_{2k+1}^0 \left(-\frac{1}{p\tau}; \chi_p \right) = \frac{\sqrt{p}}{i} \tau^{2k+1} E_{2k+1}^{\infty}(\tau; \chi_p).$$

Proof. This is a well-known result, e.g., see Cooper [11] or Kolberg [16]. □

Lemma 3.2. *For any positive integer k , we have*

$$(3.5) \quad E_{2k} \left(\tau + \frac{1}{2} \right) = -E_{2k}(\tau) + (2^{2k} + 2)E_{2k}(2\tau) - 2^{2k}E_{2k}(4\tau).$$

For $p = 3$ or 11 and any nonnegative integer k , we have

$$(3.6) \quad E_{2k+1}^0 \left(\tau + \frac{1}{2}; \chi_p \right) = -E_{2k+1}^0(\tau; \chi_p) + (2^{2k+1} - 2)E_{2k+1}^0(2\tau; \chi_p) + 2^{2k+1}E_{2k+1}^0(4\tau; \chi_p),$$

$$(3.7) \quad E_{2k+1}^{\infty} \left(\tau + \frac{1}{2}; \chi_p \right) = -E_{2k+1}^{\infty}(\tau; \chi_p) + (2 - 2^{2k+1})E_{2k+1}^{\infty}(2\tau; \chi_p) + 2^{2k+1}E_{2k+1}^{\infty}(4\tau; \chi_p).$$

For $p = 7$ or 23 and any nonnegative integer k , we have

$$(3.8) \quad E_{2k+1}^0 \left(\tau + \frac{1}{2}; \chi_p \right) = -E_{2k+1}^0(\tau; \chi_p) + (2^{2k+1} + 2)E_{2k+1}^0(2\tau; \chi_p) - 2^{2k+1}E_{2k+1}^0(4\tau; \chi_p),$$

$$(3.9) \quad E_{2k+1}^{\infty} \left(\tau + \frac{1}{2}; \chi_p \right) = -E_{2k+1}^{\infty}(\tau; \chi_p) + (2^{2k+1} + 2)E_{2k+1}^{\infty}(2\tau; \chi_p) - 2^{2k+1}E_{2k+1}^{\infty}(4\tau; \chi_p).$$

Proof. First of all, from the definitions of $E_{2k}(\tau)$, $E_{2k+1}^0(\tau; \chi_p)$ and $E_{2k+1}^{\infty}(\tau; \chi_p)$, we may deduce the Fourier expansions

$$(3.10) \quad E_{2k}(\tau) = 1 - \frac{4k}{\mathcal{B}_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

$$(3.11) \quad E_{2k+1}^0(\tau; \chi_p) = \delta_{2k+1,1} - \frac{4k+2}{\mathcal{B}_{2k+1,p}} \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{n/d}{p} \right) d^{2k} \right) q^n$$

and

$$(3.12) \quad E_{2k+1}^{\infty}(\tau; \chi_p) = 1 - \frac{4k+2}{\mathcal{B}_{2k+1,p}} \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{d}{p} \right) d^{2k} \right) q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$ if n is a positive integer. For (3.5), we first observe that

$$\sigma_k(2n) = (1 + 2^k) \sigma_k(n) - 2^k \sigma_k(n/2),$$

where $\sigma_k(n/2)$ is defined to be zero if $n/2$ is not a positive integer. Then,

$$\begin{aligned} & E_{2k} \left(\tau + \frac{1}{2} \right) + E_{2k}(\tau) \\ &= 2 \left(1 - \frac{4k}{\mathcal{B}_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(2n) q^{2n} \right) \\ &= 2 \left(1 - \frac{4k}{\mathcal{B}_{2k}} \sum_{n=1}^{\infty} \left((1 + 2^{2k-1}) \sigma_{2k-1}(n) - 2^{2k-1} \sigma_{2k-1}(n/2) \right) q^{2n} \right) \end{aligned}$$

$$\begin{aligned}
&= 2 \left(\left(1 + 2^{2k-1}\right) \left(1 - \frac{4k}{\mathcal{B}_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n}\right) - 2^{2k-1} \left(1 - \frac{4k}{\mathcal{B}_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{4n}\right) \right) \\
&= \left(2 + 2^{2k}\right) E_{2k}(2\tau) - 2^{2k} E_{2k}(4\tau).
\end{aligned}$$

If we let $\sigma_{2k+1}^0(n)$ and $\sigma_{2k+1}^\infty(n)$ be defined by

$$\sigma_{2k+1}^0(n) = \sum_{d|n} \left(\frac{n/d}{p}\right) d^{2k}, \quad \text{and} \quad \sigma_{2k+1}^\infty(n) = \sum_{d|n} \left(\frac{d}{p}\right) d^{2k},$$

then similarly, we find that

$$\sigma_{2k+1}^0(2n) = \begin{cases} (2^{2k} - 1)\sigma_{2k+1}^0(n) + 2^{2k}\sigma_{2k+1}^0(n/2), & \text{for } p = 3 \text{ or } 11, \\ (1 + 2^{2k})\sigma_{2k+1}^0(n) - 2^{2k}\sigma_{2k+1}^0(n/2), & \text{for } p = 7 \text{ or } 23, \end{cases}$$

and

$$\sigma_{2k+1}^\infty(2n) = \begin{cases} (1 - 2^{2k})\sigma_{2k+1}^\infty(n) + 2^{2k}\sigma_{2k+1}^\infty(n/2), & \text{for } p = 3 \text{ or } 11, \\ (1 + 2^{2k})\sigma_{2k+1}^\infty(n) - 2^{2k}\sigma_{2k+1}^\infty(n/2), & \text{for } p = 7 \text{ or } 23. \end{cases}$$

By the above observations, identities (3.6)–(3.9) can be proved in the same fashion, so we omit the details. \square

Lemma 3.3. *For any positive integer k , we have*

$$(3.13) \quad F_{2k} \left(-\frac{1}{4p\tau} + \frac{1}{2}; p \right) = \frac{\tau^{2k} (-16p)^k (G_{2k}(2\tau) - G_{2k}(4\tau))}{(2^{2k} - 1)(1 + (-p)^k)}.$$

For $p = 3$ or 11 and any non-negative integer k , we have

$$\begin{aligned}
(3.14) \quad & F_{2k+1} \left(-\frac{1}{4p\tau} + \frac{1}{2}; p \right) \\
&= (2i\tau\sqrt{p})^{2k+1} \times \frac{\left(2^{2k+1}G_{2k+1}(4\tau; p) - 2(2^{2k} - 1)\tilde{G}_{2k+1}(2\tau; p) - 2G_{2k+1}(\tau; p)\right)}{(2^{2k+1} + 1)(1 + \delta_{k,0})}.
\end{aligned}$$

For $p = 7$ or 23 and any non-negative integer k , we have

$$(3.15) \quad F_{2k+1} \left(-\frac{1}{4p\tau} + \frac{1}{2}; p \right) = (4i\tau\sqrt{p})^{2k+1} \times \frac{(G_{2k+1}(4\tau; p) - G_{2k+1}(2\tau; p))}{(2^{2k+1} - 1)(1 + \delta_{k,0})}.$$

Proof. These follow immediately from the definitions of $F_k(\tau; p)$ together with Lemmas 3.1 and 3.2 and the weight $2k$ modularity of E_{2k} on $\text{SL}_2(\mathbb{Z}) = \Gamma_0(1)$. \square

Now we are ready for

Proof of Theorem 2.1. Let $p = 3, 7, 11$ or 23 , and let k be a positive integer. Let ℓ be the smallest integer that satisfies

$$\ell \geq \begin{cases} \frac{(p+1)k}{8} - \frac{1}{2}, & \text{if } p = 3 \text{ or } 11 \text{ and } k \text{ is odd,} \\ \frac{(p+1)k}{8} - 1, & \text{otherwise.} \end{cases}$$

Consider the functions

$$f(\tau) = f_{k,p}(\tau) = \frac{F_k(\tau; p)}{z_p(\tau)^k x_p(\tau)^\ell} \quad \text{and} \quad g(\tau) = g_p(\tau) = \frac{1}{x_p(\tau)}.$$

Clearly, both $f(\tau)$ and $g(\tau)$ are analytic on \mathbb{H} . And we may verify that both $f(\tau)$ and $g(\tau)$ are invariant under $\Gamma_0(4p)$ and

$$W_e = \left\{ \begin{pmatrix} ae & b \\ 4pc & de \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ the determinant is } e \right\}$$

for $e \in \{4, p, 4p\}$. Therefore, both $f(\tau)$ and $g(\tau)$ are invariant under $\Gamma_0(4p)^+$, the group obtained from $\Gamma_0(4p)$ by adjoining all of its Atkin-Lehner involutions W_e . Let us analyze the behavior at $\tau = i\infty$. By observing the q -expansions, we find that

$$f(\tau) = \frac{1 + O(q)}{(1 + O(q))^k q^\ell (1 + O(q))^\ell} = q^{-\ell} + O(q^{-\ell+1}).$$

Therefore $f(\tau)$ has a pole of order ℓ at $i\infty$. Similarly, we note that $g(\tau)$ has a simple pole at $\tau = i\infty$. It implies that there exist constants $a_1, \dots, a_\ell \in \mathbb{C}$ such that the function

$$h(\tau) := f(\tau) - \sum_{j=1}^{\ell} a_j g(\tau)^j$$

has no pole at $\tau = i\infty$, that is,

$$h(\tau) = a_0 + O(q) \quad \text{as } \tau \rightarrow i\infty$$

for some constant a_0 . Let us consider the behavior of $h(\tau)$ at $\tau = \frac{1}{2}$. By Lemma 3.3 and the transformation formula for Dedekind's eta function, we find that

$$f\left(-\frac{1}{4p\tau} + \frac{1}{2}\right) = \begin{cases} C_0 q^{2\ell - \frac{(p+1)k}{4} + 1} (1 + O(q)), & \text{if } p = 3 \text{ or } 11 \text{ and } k \text{ is odd,} \\ C_1 q^{2\ell - \frac{(p+1)k}{4} + 2} (1 + O(q)), & \text{otherwise,} \end{cases}$$

and

$$g\left(-\frac{1}{4p\tau} + \frac{1}{2}\right) = C_3 q^2 (1 + O(q))$$

for some constants C_0, C_1 and C_2 as $\tau \rightarrow i\infty$. Therefore, $h(\tau) \rightarrow a_0$ as $\tau \rightarrow \frac{1}{2}$. Since the only cusps of $\Gamma_0(4p)^+$ are at $i\infty$ and $\frac{1}{2}$, it follows that $h(\tau)$ is holomorphic on $X(\Gamma_0(4p)^+)$, and thus $h(\tau)$ is a constant, that is, $h(\tau) \equiv a_0$. Therefore, we have

$$f(\tau) = \sum_{j=0}^{\ell} a_j g(\tau)^j,$$

which is equivalent to

$$F_k(\tau; p) = z_p^k \sum_{j=0}^{\ell} a_j x_p^{\ell-j} = z_p^k \sum_{j=0}^{\ell} b_j x_p^j,$$

where $b_j := a_{\ell-j}$. By the choice of ℓ and comparing the constant terms on both sides, we conclude that $b_0 = 1$ and

$$F_k(\tau; p) = z_p^k + z_p^k \sum_{1 \leq j < \frac{(p+1)k}{8}} b_j x_p^j.$$

Now take $b_j = -c_{p,k,j}$ to complete the proof. □

Proof of Corollary 2.2. It follows directly from Theorem 2.1 together with the following observations:

$$\tilde{z}_p(\tau) = \frac{i}{4\sqrt{p}\tau} z_p \left(-\frac{1}{2p\tau} + \frac{1}{2} \right),$$

$$(3.16) \quad \begin{aligned} \tilde{x}_p(\tau) &= -2^{-24/(p+1)} \tilde{x}_p \left(-\frac{1}{2p\tau} + \frac{1}{2} \right), \\ \tilde{F}_k(\tau; p) &= \left(\frac{i}{4\sqrt{p\tau}} \right)^k F_k \left(-\frac{1}{2p\tau} + \frac{1}{2} \right). \end{aligned}$$

□

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